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# THE ATTRACTION OF A RIGHT CIRCULAR CYLINDER ON A PARTICLE.

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[CONTINUED FROM VOL. II, PAGE 38.]

One of the best illustrations of the advantage of this notation is the imaginary transformation of Jacobi, which may be given because it will be used in the reduction of (4).

Assume in (5)
$$\sin\varphi = i\tan\psi,$$

or
$$\cos\varphi = \frac{1}{\cos\psi},$$

or
$$\Delta\varphi = \frac{\Delta(\psi_\beta)_{-\beta}}{\cos\psi}; \tag{7}$$

then
$$u = \int_0^\phi \frac{d\varphi}{\Delta\varphi} = i\int_0^\psi \frac{d\psi}{\Delta(\psi_\beta)_{-\beta}}, \tag{8}$$

or
$$u = \varphi_\gamma = i\psi_\beta; \tag{9}$$

$$\therefore\ u_{-\gamma} = \varphi = (i\psi_\beta)_{-\gamma}; \tag{9'}$$

also
$$\left(\frac{u}{i}\right)_{-\beta} = \left(\frac{\varphi_\gamma}{i}\right)_{-\beta} = \psi; \tag{9''}$$

therefore
$$\sin u_{-\gamma} = i\tan\left(\frac{u}{i}\right)_{-\beta} = \frac{1}{i}\tan(iu)_{-\beta}.$$

$$\cos u_{-\gamma} = \frac{1}{\cos\left(\frac{u}{i}\right)_{-\beta}} = \frac{1}{\cos(iu)_{-\beta}},$$

$$\Delta u_{-\gamma} = \frac{\Delta \left(\frac{u}{i}\right)_{-\beta}}{\cos \left(\frac{u}{i}\right)_{-\beta}} = \frac{\Delta (iu)_{-\beta}}{\cos (iu)_{-\beta}}. \tag{7'}$$

Jacobi's type for the integral of the third species is with our notation:

$$\Pi (\varphi_\gamma, a_\gamma) = \int_0^\phi \frac{\gamma^2 \sin a \cos a \, \Delta a \sin^2 \varphi \, d\varphi}{(1 - \gamma^2 \sin^2 a \sin^2 \varphi) \, \Delta\varphi}$$

$$= \frac{\Delta a}{\tan a} \left( \int_0^\phi \frac{d\varphi}{(1 - \gamma^2 \sin^2 a \sin^2 \varphi) \, \Delta\varphi} - \int_0^\phi \frac{d\varphi}{\Delta\varphi} \right)$$

$$= \frac{\Delta a}{\tan a} [\Pi_1 (\varphi, - \gamma^2 \sin^2 a)^* - \varphi_\gamma]. \tag{10}$$

Comparing this with the third term of (4) we have $\varphi = \Delta, \gamma = \frac{c_e}{a_e}$, and

$$\sin^2 a = - \frac{e^2}{(d - r)^2}. \tag{11}$$

The angle $a$ must be purely imaginary and if we assume by (7)

$$\sin a = i \tan \left(\frac{a_\gamma}{i}\right)_{-\beta} = i \tan \mathrm{A}, \text{ say, where } \mathrm{A} \text{ is real,} \tag{12}$$

then by (9′)

$$a = (i\mathrm{A}_\beta)_{-\gamma}. \tag{13}$$

We have by (11) and (12)

$$\tan \mathrm{A} = \frac{e}{d - r}; \tag{14}$$

$$\therefore \sin \mathrm{A} = \frac{e}{b_e}, \quad \cos \mathrm{A} = \frac{d - r}{b_e}, \quad \Delta (\mathrm{A}_\beta)_{-\beta} = \frac{d + r}{a_e}. \tag{15}$$

We have then in (4), regarding (10),

$$A_e = m\vartheta \frac{eb_e}{d \sin^2 \mathrm{A}} \Big\{ \Delta_\gamma - \sin^2 \mathrm{A} \, E(\Delta_\gamma)$$

$$- \Delta (\mathrm{A}_\beta)^2_{-\beta} \left[ .\Delta_\gamma + \frac{\tan (i\mathrm{A}_\beta)_{-\gamma}}{\Delta (i\mathrm{A}_\beta)_{-\gamma}} \, \Pi (\Delta_\gamma, i\mathrm{A}_\beta) \right] \Big\}$$

$$= m\vartheta \frac{eb_e}{d \sin^2 \mathrm{A}} \left[ [\beta^2 \Delta_\gamma - E(\Delta_\gamma)] \sin^2 \mathrm{A} - i \sin \mathrm{A} \cos \mathrm{A} \, \Delta (\mathrm{A}_\beta)_{-\beta} \, \Pi (\Delta_\gamma, i\mathrm{A}_\beta) \right]$$

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*Legendre's notation is used.

$$= m\vartheta\frac{eb_e}{d}\left[\beta^2\,\Delta_\gamma - E(\,\Delta_\gamma) - i\,\frac{\varDelta\,(\mathrm{A}\beta)-\beta}{\tan\mathrm{A}}\,\Pi(\,\Delta_\gamma, i\mathrm{A}_\beta)\right]. \tag{16}$$

Jacobi's theorem for exchanging amplitude with parametric angle, which is so easily remembered by its symmetry, is

$$\Pi\,(\varphi_\gamma, \alpha_\gamma) - \Pi\,(\alpha_\gamma, \varphi_\gamma) = \varphi_\gamma E\,(\alpha_\gamma) - \alpha_\gamma E\,(\varphi_\gamma). \tag{17}$$

If in this we place $\varphi = \Delta$, then since

$$\Pi\,(\alpha_\gamma, \Delta_\gamma) = \int_0^\phi \frac{\gamma^2\sin\Delta\,\cos\Delta\;\varDelta\,\Delta\,\sin^2\varphi\,d\varphi}{(1-\gamma^2\sin^2\alpha\,\sin^2\varphi)\,\varDelta\varphi} = 0,$$

we have

$$\Pi\,(\,\Delta_\gamma, \alpha_\gamma) = \,\Delta_\gamma\,E\,(\alpha_\gamma) - E\,(\,\Delta_\gamma), \tag{18}$$

and thus the quadrantal integral of the third species is in a very simple manner expressed in terms of integrals of the first and second species if $\alpha$ is real. Jacobi has shown that all possible cases of parameter are contained in his type. In the present case we have

$$\Pi\,(\,\Delta_\gamma, i\mathrm{A}_\beta) = \Delta_\gamma\,E\,(i\mathrm{A}_\beta) - i\mathrm{A}_\beta\,E\,(\,\Delta_\gamma). \tag{19}$$

It is necessary to express $E\,(i\mathrm{A}_\beta)$ in real terms.

We have $E\,(i\mathrm{A}_\beta) = \int_0^{(i\mathrm{A}_\beta)-\gamma} d\varphi\,\varDelta\varphi = i\int_0^{\mathrm{A}} \frac{d\varphi}{\varDelta\,(\varphi_\beta)-\beta}\cdot\frac{\varDelta\,(\varphi_\beta)^2-\beta}{\cos^2\varphi}$ by (8) and (7′)

$$= i\tan\mathrm{A}\;\varDelta\,(\mathrm{A}_\beta)_{-\beta} + i\int_0^{\mathrm{A}}\tan\varphi\,.\,\frac{\gamma^2\sin\varphi\cos\varphi\,d\varphi}{\varDelta\,(\varphi_\beta)-\beta}$$

$$= i\tan\mathrm{A}\;\varDelta\,(\mathrm{A}_\beta)_{-\beta} + i\mathrm{A}_\beta - iE\,(\mathrm{A}_\beta). \tag{20}$$

With this we have in (19)

$$\Pi\,(\,\Delta_\gamma, i\mathrm{A}_\beta) = i\,[\tan a\;\varDelta\,(\mathrm{A}_\beta)_{-\beta} + \mathrm{A}_\beta - E\,(\mathrm{A}_\beta)]\,\Delta_\gamma - i\mathrm{A}_\beta\,E(\,\Delta_\beta);$$

and, using this in (16), we have

$$A_e = m\vartheta\frac{eb_e}{d}\times$$

$$\left(\beta^2\,\Delta_\gamma - E(\,\Delta_\gamma) + \varDelta(\mathrm{A}_\beta)^2{}_{-\beta}\,\Delta_\gamma - \frac{\varDelta\,(\mathrm{A}_\beta)_{-\beta}.}{\tan\mathrm{A}}[\,\Delta_\gamma\mathrm{A}_\beta - \mathrm{A}_\beta E(\,\Delta_\gamma) - \,\Delta_\gamma E(\mathrm{A}_\beta)]\right)$$

$$= m\vartheta\frac{eb_e}{d}\times \tag{21}$$

$$\left((1+\beta^2\cos^2\mathrm{A})\,\Delta_\gamma - E(\,\Delta_\gamma) - \frac{\varDelta(\mathrm{A}_\beta)-\beta}{\tan\mathrm{A}}[\,\Delta_\gamma\mathrm{A}_\beta - \mathrm{A}_\beta\,E(\,\Delta_\gamma) - \,\Delta_\gamma E(\mathrm{A}_\beta)]\right)$$

This form is more lengthy than (4), but has the advantage of being expressed entirely in terms of tabulated functions. If the attracted particle is at midheight, then $e = h - e$ and the total attraction $A = 2A_e$. If $e$ is infinitesimal, we obtain the attraction of a circular plate on a particle in its plane which may be expressed

$$A = 2\left(A_0 + \frac{dA_e}{de}\delta e\right), \tag{22}$$

where $2\delta e$ = thickness of the plate. But $A_0 = 0$; hence

$$A = 2\frac{dA_e}{de}\delta e \tag{22'}$$

$$= 2m\frac{\vartheta}{d}\delta e\left\{\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{\left(\frac{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}{(\rho^2+e^2)}\right)}\right.$$

$$\left. - e^2\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{\left(\frac{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}{(\rho^2+e^2)^3}\right)}\right\}_{e=0}$$

$$= 2m\frac{\vartheta}{d}\delta e\int_{d-r}^{d+r}\frac{d\rho}{\rho^2}\sqrt{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}$$

$$\left(\rho = \frac{d-r}{\Delta\varphi};\ \Delta\varphi = \sqrt{\left(1 - \frac{4rd}{(d+r)^2}\sin^2\varphi\right)}\right)$$

$$= 2m\frac{\vartheta}{d}\delta e\,(d+r)\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi^3}(1-\Delta\varphi^2)\left[\Delta\varphi^2 - \left(\frac{d-r}{d+r}\right)^2\right]$$

$$= 2m\frac{\vartheta}{d}\delta e\,(d+r)\left[-\left(\frac{d-r}{d+r}\right)^2\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi^3} + 2\frac{d^2+r^2}{(d+r)^2}\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi} - \int_0^{\lrcorner}d\varphi\,\Delta\varphi\right]$$

$$= 4m\frac{\vartheta}{d}\delta e\left(\frac{d^2+r^2}{d+r}F - (d+r)E\right)$$

$$\left(\gamma = \frac{2\sqrt{(rd)}}{d+r};\ \beta = \frac{d-r}{d+r}\right)$$

$$= 4m\frac{\vartheta}{d}\delta e\,(d+r)\left[(1+\beta^2)\lrcorner_\gamma - E(\lrcorner_\gamma)\right]. \tag{22''}$$

This form could, by evaluation of the indeterminate part, be derived from (21), which is to be doubled first.

If we follow Price's method of determining the attraction of plates (Price's *Infinitesimal Calculus*, Vol. III, p. 275) we obtain the identical result. But Price, in example of the attraction of a rectangular plate, commits a manifest blunder in the making the attraction on a particle at either one of its corners infinite. The corresponding case here is the attraction on a particle at the rim of the plate for which $d = r$; $\therefore$ $\beta = 0$; $\gamma = 1$. Now, since

$$\int_0^{\Delta} \frac{d\varphi}{\Delta(\varphi_1)_{-1}} = \int_0^{\Delta} \frac{d\varphi}{\cos\varphi} = l \tan\tfrac{1}{2}(\Delta + \varphi)\Big]_0^{\Delta} = \infty,$$

we have also here apparently an infinite attraction, which is, however, multiplied by $\delta e$, which should be considered infinitesimal. This product, $\delta e \times \Delta_1$, could, however, not be evaluated because as it is introduced in the formula $\delta e$ is perfectly arbitrary. The attraction of a circular plate of sensible thickness, $2\delta e$, on a particle at the circumference is evidently obtained by placing in (4) $d = r$; $\therefore$ $a_e = \sqrt{(4r^2 + \delta e^2)}$; $b_e = \delta e$; $c_e = 2r$; then

$$A = 2m\vartheta\frac{\delta e}{r}\sqrt{(4r^2 + \delta e^2)}\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi^3}\left(\Delta\varphi^2 - \frac{\delta c^2}{4r^2 + \delta e^2}\right)$$

$$= 2m\vartheta\frac{\delta e}{r}\sqrt{(4r^2 + \delta e^2)}\left(\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi} - \frac{\delta c^2}{4r^2 + \delta e^2}\int_0^{\lrcorner}\frac{d\varphi}{\Delta\varphi^3}\right)$$

$$\left(\gamma = \frac{2r}{\sqrt{(4r^2 + \delta e^2)}}\right) = 2m\vartheta\frac{\delta e}{r}\sqrt{(4r^2 + \delta e^2)}\left[\lrcorner_\gamma - E(\lrcorner_\gamma)\right]. \qquad (23)$$

As long as $\delta e$ is finite, this value of the attraction of a cylinder is finite. From this it is evident already that the attraction of a plate, being necessarily less, *cannot be infinite*. But it may also be shown that the first term of (23), although appearing under an indeterminate form if $\delta e = 0$, is $= 0$. For we have

$$\left.\frac{\displaystyle\int_0^{\lrcorner}\frac{d\varphi}{\sqrt{\left(1 - \frac{4r^2}{4r^2 + \delta c^2}\sin^2\varphi\right)}}}{\frac{1}{\delta c}}\right\}_{\delta e = 0} = \left.\frac{\displaystyle\int_0^{\lrcorner} - \frac{8r^2\delta e \sin^2\varphi\, d\varphi}{(4r^2 + \delta e^2)^2\,\Delta\varphi^3}}{-\frac{1}{\delta c^2}}\right\}_{\delta e = 0} = 0.$$